ALGEBRA GENERATORS AND INFORMATION THEORY

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ABSTRACT. A general simplicity problem in category theory is proposed. A particular example, the simplest choice of generators of an algebra is specified and illustrated by an example.

The idea of physics and of some of mathematics may be seen as the expansion and compression of information about the real world or, better, of our understanding of the real world: The understanding has to be correct and simple.

But what is the correct judgement on simplicity? A vague maximum principle, Occam's razor is not always well formalized. As a concept, a measure of simplicity is at hand: The simpler the thing, the less information is needed to describe it and there is an accepted measure of information, entropy. This runs however into at least two problems:

(1) To understand the information, a context is needed. Is the background information contained in the context also to be measured? Here we solve this by clearly specifying what structure is considered background information and what additional information is measured.

(2) A simple-minded evaluation of entropy leads in cases involving infinite sets or continuous parameters to infinities. In order to avoid a technically demanding analysis, we count here the number of continuous parameters to be specified, discounting data given by finite natural numbers. At the same time we force the number of parameters to be finite.

The outline of the paper is the following: Section 1 measures the information of an object in an abelian category against a background knowledge category. Section 2 specifies to the case of algebras and sets. An explicit result of [3], Proposition 1 provides an instance of solving the general problem. A summary is given in the Conclusion.

1. THE SIMPLICITY OF FREE RESOLUTIONS

The proposal of this section is to measure the information content of a suitable mathematical structure in an abelian category according to the following principles:

- There is a category A and an abelian category B linked by a forgetful functor $U : B \to A$ and by a free functor $F : A \to B$, the adjoint of U. The objects of interest reside in category B while category A specifies the background information.
- The functors U, F are part of the background information. Their application is not considered as connected to the provision of additional information.
- The information content of an object b of the category B to be measured is the information that needs to be added to Ub in order to understand the

structure of b. This is provided in the form of a free resolution of b. A free resolution [2] is then given by the following diagram:

(1)
$$\xrightarrow{f_4} Fa_3 \xrightarrow{f_3} Fa_2 \xrightarrow{f_2} Fa_1 \xrightarrow{f_1} b$$

$$i_3 \uparrow \qquad i_2 \uparrow \qquad i_1 \uparrow$$

$$a_3 \qquad a_2 \qquad a_1$$

where the upper row is exact. The vertical arrows indicate the functor F, the mappings $i_\bullet$ should be correctly interpreted as the inclusions of generators $i_\bullet : a_\bullet \to UFa_\bullet$.

According to these rules, the information contained in the object b of the category B with respect to the background knowledge category A is contained in the maps

(2) $\qquad Uf_k \circ i_k : a_k \to UFa_{k-1} \qquad\qquad$ for $k \in \{2, 3, 4, ...\}$

However, these maps do not need to be specified entirely: There may be automorphisms of the lower row of the resolution (9) and respecting the grading of the resolution that lift to automorphisms of the upper row. Such automorphisms will be called Bogoliubov automorphisms and determine the freedom in the maps (2) that does not affect the free resolution (9) and does therefore not need to be specified. Now, different free resolutions of b will have different information contents and it is natural to assign to b the lowest possible information content attainable by a free resolution.

We may rephrase this by saying that the information content of b is the information contained in its simplest object a_1 of generators (included in b via the map $Uf_1 \circ i_1 : a_1 \to Ub$) together with its relations as given by the free resolution (9)

Thus, the information to be measured is the information contained in the maps (2) up to Bogoliubov transformations.

That this can be indeed done is shown in an example of the following section.

2. Finite free resolutions of algebras and their information content

2.1. **Finite free resolutions.** Let A be the category of vector spaces and B the category of C^*-algebras. Let b be a finite dimensional C^*-algebra. Then a linear (grading preserving) transformation transformation in the lower row of (9) can only be lifted to an algebra automorphism in the upper row if it maps generators into generators with the same spectrum. This is obviously not always the case, for any linear transformation of the lower row of generators of (9).

It is now possible to enquire into how much information is contained in a given set of generators or, in other words, in a given free resolution of the object b, an algebra. Unfortunately, the simple idea of just counting, whatever information there is and arriving at a quantitative result is hindered by the appearance of infinities which have to be dealt with properly. To avoid such technically demanding problems, we will restrict ourselves to finite dimensional algebras and to finite resolutions only.

Definition 1. A free resolution (9) of an algebra is called here a *finite free resolution* if

(1) there is only a finite number N of nonzero terms $Fa_1, Fa_2, \ldots, Fa_N$ in the resolution,

(2) the dimensions d_j of the vector spaces a_j are all finite,

(3) the maps $f_j \circ i_j : a_j \to Fa_{j-1}$, $j = 2, 3, 4, \dots$ send a_j into a subspace of Fa_{j-1} of finite degree ∂_{j-1} of the filtration of Fa_{j-1} by word length.

Remark 1. While the first two conditions of the definition of a finite free resolution are entirely straightforward, the last, third condititon deserves some further justification. The maps f_i which are an essential part of the free resolution are uniquely determined by the maps $f_j \circ i_j : a_j \to Fa_{j-1}$, $j = 2, 3, 4, \dots$ and the information contained in them is to be included into the information contained in the free resolution. In fact, this is the dominant information content of the resolution. Unless a_{j-1} is of dimension $d_{j-1} = 0$, the free algebra Fa_{j-1} is necessarily infinite dimensional, spanned as a linear space by (finite) words with letters being elements in a_j and is canonically a filtered algebra. The linear subspaces $\mathcal{F}_n Fa_{j-1}$ of this filtration of the free algebra Fa_{j-1} are spanned by words containing at most n letters.

That the range of $f_j \circ i_j : a_j \to Fa_{j-1}$ is inside is a finite subspace $\mathcal{F}_{\partial_{j-1}} Fa_{j-1}$ is a simple consequence of the finite dimension d_j of a_j and of the fact that only finite sums of words are contained in Fa_{j-1}. Thus the third requirement is not restrictive but rather serves to define the numbers ∂_j and to emphasise that infinite series are not included in the resolution.

A restatement of this may be that all relations (relations of relations, ...) of the resolution ought to be generated from finite polynomial relations (relations of relations, ...).

2.2. **The information content of a finite free resolution.** To summarize, a finite free resolution is given by the following data:

- a number N giving the length of the resolution,
- the numbers $d_1, \dots, d_N$ giving the dimensions of the vector spaces $a_1, \dots, a_N$,
- the grading numbers $\partial_1, \dots, \partial_{N-1}$,
- the algebra homomorphisms $f_1, \dots, f_N$ determined uniquely by the linear maps

$$(3) \qquad f_2 \circ i_2 : a_2 \to \mathcal{F}_{\partial_1} Fa_1$$

$$(4) \qquad f_3 \circ i_3 : a_3 \to \mathcal{F}_{\partial_2} Fa_2$$

$$(5) \qquad f_4 \circ i_4 : a_4 \to \mathcal{F}_{\partial_3} Fa_3$$

$$\vdots$$

$$(6) \qquad f_N \circ i_N : a_N \to \mathcal{F}_{\partial_{N-1}} Fa_{N-1}$$

In agreement with [1], the entropy of a natural number N will be taken to be

$$(7) \qquad S_N = \ln N,$$

and the entropy of all numbers in the data of a finite free resolution is just the sum of the entropies of all numbers:

$$(8) \qquad S_{numbers} = \ln N + \sum_{j=1}^{N} \ln d_j + \sum_{j=1}^{N-1} \ln \partial_j.$$

It remains now to find the entropy of the maps $f_N \circ i_N : a_j \to \mathcal{F}_{\partial_{N-1}} Fa_{N-1}$ modulo Bogoljubov automorphisms. That is the essential part of the entropy calculation, since specifying these maps requires to determine real parameters. These

contain an infinite number of digits and carry therefore an infinite entropy against which one typically can neglect $S_{numbers}$. Thus the minimalization of required information boils down to finding the set of maps (3)-(6) containing up to Bogoliubov automorphisms the least number of continuous parameter specifications.

This problem has been solved in [3] for the case of b being the algebra $M_2(\mathbb{C})$ of 2×2-matrices with complex numbers in its entries. The solution is found to be the Clifford resolution of $M_2(\mathbb{C})$:

Definition 2. Let $M_{2^m}(\mathbb{C})$ be the algebra of $2^m \times 2^m$-matrices. Its **Clifford resolution** is the following finite free resolution:

$$(9) \qquad 0 \longrightarrow Fa_2 \xrightarrow{f_2} Fa_1 \xrightarrow{\gamma} M_{2^m}(\mathbb{C})$$

$$\begin{array}{cc} i_2 \uparrow & i_1 \uparrow \\ V_{m(2m+1)} & V_{2m} \end{array}$$

Where V_{2m} is the $2m$-dimensional complex vector space of generators with a basis e_1, e_2, ..., e_{2m} and where $V_{m(2m+1)}$ is the $m(2m+1)$-dimensional complex vector space of relations with a basis $r_{1\leq 1}$, $r_{1\leq 2}$, ..., $r_{2m\leq 2m}$ mapped by $f_2 \circ i_2$ onto the relations

$$(10) \qquad f_2 \circ i_2 r_{k\leq l} = e_k e_l + e_l e_k - \delta_{kl} \qquad \text{for } k, l \in \{1, 2, ..., 2m\} \text{ and } k \leq l$$

Proposition 1. *The finite free resolution of the matrix algebra $M_2(\mathbb{C})$ with the least number of parameters to be specified is given by the Clifford resolution.*

For the convenience of the reader we provide the following

Proof. Since the algebra $M_2(\mathbb{C})$ is noncommutative, its subspace of generators has to be at least 2-dimensional.

The word length of relations has to be at least 2, again to avoid commutativity.

The algebra $M_2(\mathbb{C})$ is 4-dimensional, with the 2-dimensional set of generators and the unit being linearly independent, leaving thus one dimension for one linearly independent linear combination of four words of length 2. Thus at least 3 relations are required.

The Clifford resolution allows for a 1-dimensional Bogoliubov automorphism. This can be checked to be the maximal dimension for any possible Bogoliubov automorphism. □

3. Conclusion

The general proposal of Section 1 to measure information content of the object of a category through its simplest free resolution against a category of background knowledge was specified in Section 2 for the case of C^*-algebras and their linear subspaces of generators. In the simplest case of 2×2-matrices, the simplest finite free resolution is known [3]. That result adds as an example on substance to the general framework explained here and poses at the same time the challenge of more such examples.

Note that the simplest finite free resolution is of interest not just as a tool to measure information but also for providing a suitable set of generators well adapted to the algebra in question.

The case of $n \times n$-matrices will be discussed in further work of one of the authors (R. Otáhalová).

4. Acknowledgements

References

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